Chapter 9

SUPER COHERENT STATES, BOSON-FERMION REALIZATIONS AND REPRESENTATIONS OF SUPERALGEBRAS

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Abstract

Super coherent states are useful in the explicit construction of representations of superalgebras and quantum superalgebras. In this contribution, we describe how they are used to construct (quantum) boson-fermion realizations and representations of (quantum) superalgebras. We work through a few examples: $osp(1|2)$ and its quantum version $U_t[osp(1|2)]$, $osp(2|2)$ in the non-standard and standard bases and $gl(2|2)$ in the non-standard basis. We obtain free boson-fermion realizations of these superalgebras. Applying the boson-fermion realizations, we explicitly construct their finite-dimensional representations. Our results are expected to be useful in the study of current superalgebras and their corresponding conformal field theories.

1 Introduction

Supersymmetry concepts were used to provide a unified description of mixed quantum mechanical systems of bosons and fermions. The generators of supersymmetry transformations form a superalgebra. Superalgebras and their corresponding conformal field theories (CFTs) have recently attracted considerable attention in mathematical physics community, because of their applications in areas such as topological field theory [1, 2], logarithmic CFTs (see e.g. the review article [3] and references therein) and the supersymmetric method [4] to disordered systems [5, 6, 7, 8, 9, 10, 11].

Unlike ordinary bosonic algebras, there are two types of representations for most superalgebras. They are the so-called typical and atypical representations. The typical representations are irreducible and are similar to the usual representations appeared in ordinary bosonic algebras. The atypical representations have no counterpart in the bosonic algebra representations. They can be irreducible or not fully reducible (i.e. reducible or indecomposable). This makes the study of representations of superalgebras very difficult.

In most physical applications, one needs the explicit construction of finite-dimensional representations of a superalgebra. This is particularly the case in superalgebra CFTs. To

construct primary fields of such CFTs in terms of free fields, one has to construct the finite-dimensional representations of the superalgebras explicitly. The explicit construction of primary fields is essential in the investigation of disordered systems by the supersymmetric method.

As is well-known, coherent states played an important role in a wide variety of quantum mechanical problems and, in particular, in the explicit construction of representations of symmetry algebras. Super coherent states associated with superalgebras were introduced in [12, 13]. So it is natural to investigate representations of superalgebras by means of super coherent states.

In the present contribution, we describe how the super coherent state method can be used to explicitly construct finite-dimensional representations of superalgebras. In this approach, the first step is to obtain free boson-fermion realization of a superalgebra. Then representations are constructed out of particle states in a super-Fock space. We work through a few examples: the rank 1 superalgebra $osp(1|2)$ and its quantum version $U_t[osp(1|2)]$, the rank 2 superalgebra $osp(2|2)$ in the non-standard and standard bases and the rank 3 non-semisimple superalgebra $gl(2|2)$ in the non-standard basis.

2 Boson-Fermion Realizations and Representations of $osp(1|2)$ and $U_t[osp(1|2)]$

In this section we work through two rank-one superalgebras, $osp(1|2)$ and its quantized version $U_t[osp(1|2)]$, to illustrate how to use the vector coherent state approach to construct representations of superalgebras. Generalization to higher rank cases will be disscussed in the following sections.

The superalgebra $osp(1|2)$ is generated by the odd simple elements e, f and even non-simple elements E, F. These elements obey the following (anti-)commutation relations:

$$\{e,f\}=H, \quad [H,e]=e, \quad [H,f]=-f, \quad \{e,e\}=2E, \quad \{f,f\}=-2F,$$
$$[E,F]=H, \quad [H,E]=2E, \quad [H,F]=-2F, \quad [e,F]=f, \quad [E,f]=-e. \tag{2.1}$$

The Casimir operator of this superalgebra is

$$C_2=\frac{1}{2}H(H+1)+fe+2FE. \tag{2.2}$$

Before we proceed to the vector coherent state method, we write down the Zassenhaus formula for two operators X and Y, which will be used frequently in the following. The general Zassenhaus formula reads

$$\exp(X+Y)=\exp(X)\exp(Y)\prod_{i=2}^{\infty}\exp(C_i), \tag{2.3}$$

where C_i is a homogeneous polynomial of order i in X and Y and all C_i's contain the commutator $[Y,X]$. The C_i's up to sixth order have been determined in [14]. For our purpose, we only need C_i's up to fourth order. They are given by

$$\begin{aligned} C_2 &= \frac{1}{2}[Y,X], \quad C_3 = \frac{1}{3}[[Y,X],Y] + \frac{1}{6}[[Y,X],X], \\ C_4 &= \frac{1}{8}\left([[[Y,X],Y],Y] + [[[Y,X],X],Y]\right) + \frac{1}{24}[[[Y,X],X],X]. \end{aligned} \tag{2.4}$$

Let $|hw>$ be the highest weight state of $osp(1|2)$ with highest weight j being a non-negative integer or half-integer, i.e.

$$H|hw>= 2j|hw>, \quad e|hw>= E|hw>= 0. \tag{2.5}$$

Define vector coherent states by $e^{f\alpha+aF}|hw\ >$, where α $(\alpha^\dagger)$ is a fermion destroying (creating) operator with number operator N_α and a $(a^\dagger)$ is boson annhilation (creation) operator with number operator N_a. They obey relations

$$\begin{aligned} &\{\alpha,\alpha^\dagger\} = 1, \quad [N_\alpha,\alpha] = -\alpha, \quad [N_\alpha,\alpha^\dagger] = \alpha^\dagger, \quad \alpha^2 = \left(\alpha^\dagger\right)^2 = 0, \\ &[a,a^\dagger] = 1, \quad [N_a,a] = -a, \quad [N_a,a^\dagger] = a^\dagger. \end{aligned} \tag{2.6}$$

Then state vectors $|\psi>$ are mapped into functions

$$\psi_j =< hw|e^{\alpha^\dagger e + a^\dagger E}|\psi>|0>=< hw|e^{\alpha^\dagger e}e^{a^\dagger E}|\psi>|0>. \tag{2.7}$$

Here the vacuum vector $|0>$ is defined by $\alpha|0>= a|0>= 0$.

Operators A are mapped as follows:

$$A|\psi>\rightarrow \Gamma(A)\psi_j =< hw|e^{\alpha^\dagger e}e^{a^\dagger E}A|\psi>|0>. \tag{2.8}$$

Taking $H, e, f, \cdots$ in turn we find

$$\begin{aligned} \Gamma(H) &= 2j - N_\alpha - 2N_a, \quad \Gamma(e) = \alpha + \alpha^\dagger a, \quad \Gamma(f) = 2j\alpha^\dagger - a^\dagger\alpha - \alpha^\dagger N_a, \\ \Gamma(E) &= a, \quad \Gamma(F) = a^\dagger\left(2j - N_a - N_\alpha\right). \end{aligned} \tag{2.9}$$

This gives rise to a boson-fermion realization of $osp(1|2)$. In this realization, the Casimir takes a constant value, $C_2 = 2j(j+1/2)$.

The representations of $osp(1|2)$ are constructed as follows by using the boson-fermion realization in the super-Fock space. First note that there two independent combinations of the creation operators on the vacuum vector $|0>$: $\left(a^\dagger\right)^{j-m}|0>,\ \ j-m \in \mathbf{Z}_+$ and $\alpha^\dagger\left(a^\dagger\right)^{j-m-1/2}|0>,\ \ j-m-1/2 \in \mathbf{Z}_+$. This implies that each representation of $osp(1|2)$ decomposes into at most two representations of its even subalgebra $su(2)$. It is

easy to check these two states are already representations of $su(2)$ with higest weights j and $j-1/2$, respectively. This justifies the use of notation $|j,m>$ and $|j-1/2,m>$ for the two $su(2)$ multiplets, respectively:

$$\begin{aligned} |j,m> &= \left(a^\dagger\right)^{j-m}|0>, \quad m=j,j-1,\cdots,-j, \\ |j-\frac{1}{2},m> &= \alpha^\dagger\left(a^\dagger\right)^{j-m-\frac{1}{2}}|0>, \quad m=j-\frac{1}{2},j-\frac{3}{2},\cdots,-(j-\frac{1}{2}). \end{aligned} \tag{2.10}$$

These two $su(2)$ multiplets span a spin j representation of $osp(1|2)$, to be denoted by π_j. The actions of the odd generators on the multiplets (2.10) can be easily worked out, which are not written down here. We denote by σ_j and $\sigma_{j-1/2}$ the two $su(2)$ representations in (2.10), respectively. Then the $osp(1|2)\downarrow su(2)$ branching rule is

$$\pi_j=\sigma_j\oplus\sigma_{j-1/2}. \tag{2.11}$$

The dimension of π_j is $(2j+1)+2j=4j+1$. All representations of $osp(1|2)$ are typical.

Now we generalize the method to quantum superalgebra $U_t[osp(1|2)]$. The relations satisfied by the generators of $U_t[osp(1|2)]$ are given by

$$\begin{aligned} &\{e,f\}=\frac{t^H-t^{-H}}{t-t^{-1}}, \quad [H,e]=e, \quad [H,f]=-f, \\ &\{e,e\}=2E, \quad \{f,f\}=-2F, \quad [H,E]=2E, \quad [H,F]=-2F, \\ &[E,F]=ef\frac{t^{H-1/2}+t^{-H+1/2}}{t^{1/2}+t^{-1/2}}+fe\frac{t^{H+1/2}+t^{-H-1/2}}{t^{1/2}+t^{-1/2}}, \\ &[e,F]=f\frac{t^{H-1/2}+t^{-H+1/2}}{t^{1/2}+t^{-1/2}}, \quad [E,f]=-e\frac{t^{H+1/2}+t^{-H-1/2}}{t^{1/2}+t^{-1/2}}. \end{aligned} \tag{2.12}$$

Note that the right hand side of the 3rd equation is not equal to $\frac{t^H-t^{-H}}{t-t^{-1}}$ but it degenerates to H when $t=1$. It follows that the $su(2)$ subalgebra of $osp(1|2)$ is not deformed into the usual quantum algebra $U_t[su(2)]$. Thus the usual $U_t[su(2)]$ is not a subalgebra of $U_t[osp(1|2)]$. This is the case for most superalgebras.

Define t-fermion operators $\alpha,\alpha^\dagger$ and t-boson operators $a,a^\dagger$ with number operators N_α and N_a, respectively. They satisfy the following relations:

$$\begin{aligned} &\alpha\alpha^\dagger+t\alpha^\dagger\alpha=t^{N_\alpha}, \quad [N_\alpha,\alpha]=-\alpha, \quad [N_\alpha,\alpha^\dagger]=\alpha^\dagger, \\ &\alpha^2=0=\left(\alpha^\dagger\right)^2, \quad \alpha^\dagger\alpha=[N_\alpha]_t, \quad \alpha\alpha^\dagger=[1-N_\alpha]_t, \\ &aa^\dagger-ta^\dagger a=t^{-N_a}, \quad [N_a,a]=-a, \quad [N_a,a^\dagger]=a^\dagger, \\ &a^\dagger a=[N_a]_t, \quad aa^\dagger=[N_a+1]_t, \end{aligned} \tag{2.13}$$

where

$$[x]_t\equiv\frac{t^x-t^{-x}}{t-t^{-1}}. \tag{2.14}$$

Then state vectors are mapped into functions:

$$\psi_{t,j} =< hw|\exp(\alpha^\dagger e)\exp_t(a^\dagger E)|\psi > |0 >, \tag{2.15}$$

where $\exp_t(x) \equiv \sum_{n=0}^{\infty} \frac{x^n}{[n]_t!}$ is the t-exponential, and operators A are mapped as

$$A|\psi > \to \Gamma(A)\psi_{t,j} =< hw|\exp(\alpha^\dagger e)\exp_t(a^\dagger E)A|\psi > |0 > . \tag{2.16}$$

By means of the following t-identities,

$$\begin{aligned} a\left(a^\dagger\right)^n &= t^n\left(a^\dagger\right)^n a + [n]_t\left(a^\dagger\right)^{n-1} t^{-N_a}, \\ E^n f &= fE^n - [n]_t\, eE^{n-1}\frac{t^{H+n-1/2}+t^{-H-n+1/2}}{t^{1/2}+t^{-1/2}}, \end{aligned} \tag{2.17}$$

we find from (2.16) a t-boson-fermion realization of $U_t[osp(1|2)]$:

$$\begin{aligned} \Gamma(H) &= 2j - N_\alpha - 2N_a, \quad \Gamma(e) = \alpha + \alpha^\dagger a, \\ \Gamma(f) &= [2j]_t\alpha^\dagger - \{4j-2N_a+1\}_t a^\dagger\alpha - \{4j-2N_a-1\}_t\alpha^\dagger[N_a]_t, \\ \Gamma(E) &= \{2N_\alpha - 1\}_t a, \\ \Gamma(F) &= \{4j-2N_a+1\}_t\, a^\dagger\left([2j]_t\{2N_\alpha-1\}_t - [N_a]_t\{4j-2N_a-1\}_t[1-N_\alpha]_t\right. \\ &\quad \left. -[N_a+1]_t\{4j-2N_a-3\}_t[N_\alpha]_t\right), \end{aligned} \tag{2.18}$$

where

$$\{x\}_t \equiv \frac{t^{\frac{x}{2}}+t^{-\frac{x}{2}}}{t^{\frac{1}{2}}+t^{-\frac{1}{2}}}. \tag{2.19}$$

It can be checked that (2.18) indeed satisfies the defining relations of $U_t[osp(1|2)]$.

Let us remark that boson-fermion models for $osp(1|2)$ and $U_t[osp(1|2)]$ have been obtained in [15] by using two bosons and one fermion. Our realizations given above are different as they only involve one boson and one fermion.

Representations of $U_t[osp(1|2)]$ may be constructed by means of the $t-$boson-fermion realization (2.18). The two multiplets for the unusual deformation of the even subalgebra $su(2)$ of $osp(1|2)$ have the same form as (2.10) but now $a^\dagger$ ($\alpha^\dagger$) is the $t-$ boson (fermion) operator. This is easily seen by the actions of $\Gamma(E)$ and $\Gamma(F)$ on the two multiplets:

$$\begin{aligned} \Gamma(E)|j,m> &= [j-m]_t\,|j,m+1>, \\ \Gamma(F)|j,m> &= [j+m]_t\{2(j+m-\frac{1}{2})\}_t\{2(j-m+\frac{1}{2})\}_t\,|j,m-1>, \\ \Gamma(E)|j-\frac{1}{2},m> &= [j-m-\frac{1}{2}]_t\,|j-\frac{1}{2},m+1>, \\ \Gamma(F)|j-\frac{1}{2},m> &= [j+m-\frac{1}{2}]_t\{2(j+m-\frac{1}{2})\}_t\{2(j-m+1)\}_t\,|j-\frac{1}{2},m-1> . \end{aligned} \tag{2.20}$$

The actions of the odd generators on the two multiplets are given by

$$\begin{aligned}
\Gamma(e)|j,m> &= [j-m]_t\,|j-\frac{1}{2},m+\frac{1}{2}>,\\
\Gamma(f)|j,m> &= [j+m]_t\{2(j-m+\frac{1}{2})\}_t\;|j-\frac{1}{2},m-\frac{1}{2}>,\\
\Gamma(e)|j-\frac{1}{2},m> &= |j,m+\frac{1}{2}>,\\
\Gamma(f)|j-\frac{1}{2},m> &= -\{2(j+m)\}_t\,|j,m-\frac{1}{2}>\,.
\end{aligned} \tag{2.21}$$

3 Boson-Fermion Realizations of $osp(2|2)$

Superalgebra $osp(2|2)$ can be written as $osp(2|2)=osp(2|2)^{\rm even}\oplus osp(2|2)^{\rm odd}$, with

$$osp(2|2)^{\rm even}=u(1)\oplus su(2)=\{H'\}\oplus\{H,E,F\},\quad osp(2|2)^{\rm odd}=\{e,f,\bar{e},\bar{f}\}, \tag{3.22}$$

where $e,\ f,\ \bar{e},\ \bar{f}$ are the generators corresponding to the fermionic roots, and $E,\ F$ are those to the bosonic roots. It is well-known that unlike a purely bosonic algebra a superalgebra admits different Weyl inequivalent choices of simple root systems, which correspond to inequivalent Dynkin diagrams. In the case of $osp(2|2)$, one has two choices of simple root systems which are unrelated by Weyl transformations: a system of fermionic and bosonic simple roots (i.e. the so-called standard basis), or a purely fermionic system of simple roots (that is the so-called non-standard basis). So it is useful to obtain results in the two different bases for different physical applications (see e.g. the discussions in [16, 17, 20]).

3.1 Realization in the Non-standard Basis

In the non-standard basis, simple roots of $osp(2|2)$ are all fermionic. Let $e,\ f,\ \bar{e},\ \bar{f}$ be the generators corresponding to such fermionic simple roots, and let $E,\ F$ be the non-simple generators. They obey the (anti-) commutation relations:

$$\begin{aligned}
&\{e,f\}=-\frac{1}{2}(H-H'),\quad [H,e]=e,\quad [H,f]=-f,\\
&[H',e]=e,\quad [H',f]=-f,\\
&\{\bar{e},\bar{f}\}=-\frac{1}{2}(H+H'),\quad [H,\bar{e}]=\bar{e},\quad [H,\bar{f}]=-\bar{f},\\
&[H',\bar{e}]=-\bar{e},\quad [H',\bar{f}]=\bar{f},\\
&\{e,\bar{e}\}=E,\quad \{\bar{f},f\}=-F,\\
&[E,F]=H,\quad [H,E]=2E,\quad [H,F]=-2F,\\
&[E,f]=\bar{e},\quad [F,e]=\bar{f},\quad [E,\bar{f}]=e,\quad [F,\bar{e}]=f.
\end{aligned} \tag{3.23}$$

All other (anti-)commutators are zero, and the quadratic Casimir is

$$C_2 = \frac{1}{2}\left(H^2 - H'^{\,2}\right) - 2fe - 2\bar{f}\bar{e} + 2FE. \tag{3.24}$$

Let $|hw>$ be the highest weight state of $osp(2|2)$ in the non-standard basis and (J, q) be the corresponding highest weight. Namely,

$$H|hw>= 2J|hw>, \quad H'|hw>= 2q|hw>, \quad E|hw>= e|hw>= \bar{e}|hw>= 0. \tag{3.25}$$

Define the vector coherent states, $e^{Fa+f\alpha_1+\alpha_2\bar{f}}|hw>$. Then state vectors $|\psi>$ are mapped into functions

$$\begin{aligned} \psi_{J,q} &= <hw|\exp\left(\alpha_1^\dagger e + \alpha_2^\dagger \bar{e} + a^\dagger E\right)|\psi>|0> \\ &= <hw|e^{(a^\dagger + \frac{1}{2}\alpha_1^\dagger\alpha_2^\dagger)E} e^{\alpha_1^\dagger e} e^{\alpha_2^\dagger \bar{e}}|\psi>|0>. \end{aligned} \tag{3.26}$$

Here $a, a^\dagger$ are bosonic operators with number operator N_a, and $\alpha_1\ (\alpha_1^\dagger), \alpha_2\ (\alpha_2^\dagger)$ are fermionic operators with number operators $N_{\alpha_1}, N_{\alpha_2}$, respectively. These operators satisfy relations:

$$\begin{aligned} &[a, a^\dagger] = 1, \quad [N_a, a^\dagger] = a^\dagger, \quad [N_a, a] = -a, \\ &\{\alpha_1, \alpha_1^\dagger\} = 1, \quad [N_{\alpha_1}, \alpha_1^\dagger] = \alpha_1^\dagger, \quad [N_{\alpha_1}, \alpha_1] = -\alpha_1, \\ &\{\alpha_2, \alpha_2^\dagger\} = 1, \quad [N_{\alpha_2}, \alpha_2^\dagger] = \alpha_2^\dagger, \quad [N_{\alpha_2}, \alpha_2] = -\alpha_2, \end{aligned} \tag{3.27}$$

all other (anti-)commutators are zero. Moreover, $a|0>= \alpha_1|0>= \alpha_2|0>= 0$.

Operators A in the non-standard basis are mapped as before:

$$A|\psi>\rightarrow \Gamma(A)\psi_{J,q} =<hw|e^{(a^\dagger + \frac{1}{2}\alpha_1^\dagger\alpha_2^\dagger)E} e^{\alpha_1^\dagger e} e^{\alpha_2^\dagger \bar{e}} A|\psi>|0>. \tag{3.28}$$

Taking $A = H, H', e, \cdots$ in turn we find

$$\begin{aligned} \Gamma(H) &= 2J - 2N_a - N_{\alpha_1} - N_{\alpha_2}, \quad \Gamma(H') = 2q - N_{\alpha_1} + N_{\alpha_2}, \\ \Gamma(e) &= \alpha_1 + \frac{1}{2}\alpha_2^\dagger a, \quad \Gamma(f) = -(J-q)\alpha_1^\dagger + a^\dagger\alpha_2 + \frac{1}{2}\alpha_1^\dagger\left(N_a + N_{\alpha_2}\right), \\ \Gamma(\bar{e}) &= \alpha_2 + \frac{1}{2}\alpha_1^\dagger a, \quad \Gamma(\bar{f}) = -(J+q)\alpha_2^\dagger + a^\dagger\alpha_1 + \frac{1}{2}\alpha_2^\dagger\left(N_a + N_{\alpha_1}\right), \\ \Gamma(E) &= a, \quad \Gamma(F) = 2Ja^\dagger + q\alpha_1^\dagger\alpha_2^\dagger - a^\dagger\left(N_a + N_{\alpha_1} + N_{\alpha_2}\right). \end{aligned} \tag{3.29}$$

This gives a free boson-fermion realization of $osp(2|2)$ in the non-standard basis. In this realization, the Casimir takes a constant value, i.e. $C_2 = 2(J^2 - q^2)$.

3.2 Realization in the Standard Basis

Let E (F) and e (f) be the generators corresponding to the even and odd simple roots of $osp(2|2)$ in the standard (distinguished) basis, respectively. Let $\bar{e},\ \bar{f}$ be the odd non-simple generators. They satisfy the following (anti-)commutation relations:

$$\begin{aligned}
&[E,F]=H, \quad [H,E]=2E, \quad [H,F]=-2F,\\
&\{e,f\}=-\frac{1}{2}(H-H'), \quad [H,e]=-e, \quad [H,f]=f,\\
&[H',e]=-e, \quad [H',f]=f,\\
&[E,e]=\bar{e}, \quad [F,f]=\bar{f},\\
&\{\bar{e},\bar{f}\}=-\frac{1}{2}(H+H'), \quad [H,\bar{e}]=\bar{e}, \quad [H,\bar{f}]=-\bar{f},\\
&[H',\bar{e}]=-\bar{e}, \quad [H',\bar{f}]=\bar{f},\\
&\{e,\bar{f}\}=-F, \quad \{\bar{e},f\}=E, \quad [E,\bar{f}]=f, \quad [F,\bar{e}]=e.
\end{aligned} \tag{3.30}$$

All other (anti-)commutators are zero. The quadratic Casimir is given by

$$C_2=\frac{1}{2}\left(H(H+2)-H'(H'+2)\right)+2fe-2\bar{f}\bar{e}+2FE. \tag{3.31}$$

Let $|hw>$ be the highest weight state of highest weight (p,q) of $osp(2|2)$ in the standard basis:

$$H|hw>=2p|hw>, \quad H'|hw>=2q|hw>, \quad E|hw>=e|hw>=\bar{e}|hw>=0. \tag{3.32}$$

Then similar to the non-standard basis case, state vectors $|\psi>$ in the standard basis are mapped into functions

$$\begin{aligned}
\psi_{p,q} &= <hw|\exp\left(a^\dagger E+\alpha_1^\dagger e+\alpha_2^\dagger\bar{e}\right)|\psi>|0>\\
&= <hw|e^{a^\dagger E}e^{\alpha_1^\dagger e}e^{(\alpha_2^\dagger-\frac{1}{2}\alpha_1^\dagger a^\dagger)\bar{e}}|\psi>|0>,
\end{aligned} \tag{3.33}$$

and operators A are mapped as before:

$$A|\psi>\longrightarrow \Gamma(A)\psi_{J,q}=<hw|e^{a^\dagger E}e^{\alpha_1^\dagger e}e^{(\alpha_2^\dagger-\frac{1}{2}\alpha_1^\dagger a^\dagger)\bar{e}}A|\psi>|0>. \tag{3.34}$$

After some algebraic manipulations, we find

$$\begin{aligned}
\Gamma(H) &= 2p-2N_a+N_{\alpha_1}-N_{\alpha_2}, \quad \Gamma(H')=2q+N_{\alpha_1}+N_{\alpha_2},\\
\Gamma(E) &= a-\frac{1}{2}\alpha_1^\dagger\alpha_2,\\
\Gamma(F) &= 2pa^\dagger-\alpha_2^\dagger\alpha_1-a^\dagger\left(N_a-\frac{1}{2}N_{\alpha_1}+\frac{1}{2}N_{\alpha_2}\right)-\frac{1}{4}(a^\dagger)^2\alpha_1^\dagger\alpha_2,
\end{aligned}$$

$$\begin{aligned}\Gamma(e) &= \alpha_1 + \frac{1}{2}a^\dagger\alpha_2, \quad \Gamma(f) = -(p-q)\alpha_1^\dagger + \alpha_2^\dagger a + \frac{1}{2}\alpha_1^\dagger\left(N_a + N_{\alpha_2}\right), \\ \Gamma(\bar{e}) &= \alpha_2, \quad \Gamma(\bar{f}) = -(p+q)\alpha_2^\dagger - \frac{1}{2}(3p-q)a^\dagger\alpha_1^\dagger + \alpha_2^\dagger\left(N_a - N_{\alpha_1}\right) + \frac{1}{2}a^\dagger\alpha_1^\dagger N_a.\end{aligned} \tag{3.35}$$

This is the free boson-fermion realization of $osp(2|2)$ in the standard basis. In this realization, $C_2 = 2[p(p+1) - q(q+1)]$.

4 Construction of Representations of $osp(2|2)$

We now use the above free boson-fermion realizations to construct finite-dimensional representations of $osp(2|2)$ in both the non-standard and standard bases. As we will see, all finite-dimensional typical and atypical representations of $osp(2|2)$ can be constructed in an unified manner.

4.1 Representations in the Non-standard Basis

Representations of $osp(2|2)$ in the non-standard basis were investigated in [18, 19]. Here we reproduce those results by means of the free boson-fermion realization obtained in the previous section. Our method is different from, and in our view simpler than, the methods used in [18, 19]. To begin with, we note that representations of $osp(2|2)$ in the non-standard basis are labelled by (J, q), where J is a non-negative integer or half-integer and q is an arbitrary complex number. There are four independent combinations of creation operators acting on the vacuum vector $|0>$:

$$\begin{aligned}&\left(a^\dagger\right)^{J-m}|0>, \quad J-m \in \mathbf{Z}_+, \\ &\alpha_1^\dagger\left(a^\dagger\right)^{J-m-1/2}|0>, \quad J-\frac{1}{2}-m \in \mathbf{Z}_+, \\ &\alpha_2^\dagger\left(a^\dagger\right)^{J-m-1/2}|0>, \quad J-\frac{1}{2}-m \in \mathbf{Z}_+, \\ &\alpha_1^\dagger\alpha_2^\dagger\left(a^\dagger\right)^{J-m-1}|0>, \quad J-1-m \in \mathbf{Z}_+.\end{aligned} \tag{4.36}$$

Thus each $osp(2|2)$ representation decomposes into at most four representations of the even subalgebra $su(2)\oplus u(1)$. Let us construct representations for $su(2)\oplus u(1)$ out of the above particle states in the super-Fock space. It is easy to check that the last three states are already representations of $su(2)\oplus u(1)$ with highest weights $(J-\frac{1}{2}, q-\frac{1}{2})$, $(J-\frac{1}{2}, q+\frac{1}{2})$ and $(J-1, q)$, respectively. This justifies the use of the notations, $|J-\frac{1}{2}, m; q-\frac{1}{2}>$, $|J-\frac{1}{2}, m; q+\frac{1}{2}>$ and $|J-1, m; q>$ for these three multiplets, respectively. To turn the first state into a representation of $su(2)\oplus u(1)$, we make the following ansatz in view

of the free boson-fermion expressions of generators $\Gamma(H'),\Gamma(H),\Gamma(E),\Gamma(F)$,

$$\eta^m_{J,q}=r^m_{J,q}\left(a^\dagger\right)^{J-m}|0>+\bar{r}^m_{J,q}\alpha^\dagger_1\alpha^\dagger_2\left(a^\dagger\right)^{J-m-1}|0>, \tag{4.37}$$

where $r^m_{J,q},\bar{r}^m_{J,q}$ are functions of J,q,m to be determined. It is easily shown that $\Gamma(H)\eta^m_{J,q}=2m\eta^m_{J,q}$ and $\Gamma(H')\eta^m_{J,q}=2q\eta^m_{J,q}$. So this state has highest weight (J,q) and thus will be written as $|J,q;m>$ in the following. Now

$$\Gamma(E)|J,m;q>=(J-m)r^m_{J,q}\left(a^\dagger\right)^{J-(m+1)}|0>+(J-m-1)\bar{r}^m_{J,q}\alpha^\dagger_1\alpha^\dagger_2\left(a^\dagger\right)^{J-(m+1)-1}|0>. \tag{4.38}$$

The r.h.s. must equal to $(J-m)|J,m+1;q>$ for the representation to be finite-dimensional. It follows that

$$\begin{aligned} r^m_{J,q}=r^{m+1}_{J,q} &\implies r^m_{J,q}=c_{J,q},\\ (J-m)\bar{r}^{m+1}_{J,q}=(J-m-1)\bar{r}^m_{J,q} &\implies \bar{r}^m_{J,q}=(J-m)\bar{c}_{J,q}, \end{aligned} \tag{4.39}$$

where $c_{J,q},\bar{c}_{J,q}$ are functions of J,q only. Finally the action of $\Gamma(F)$ on $|J,q;m>$ gives

$$\begin{aligned} \Gamma(F)|J,m;q> &= c_{J,q}(J+m)\left(a^\dagger\right)^{J-(m-1)}|0>\\ &+ [c_{J,q}q+\bar{c}_{J,q}(J-m)(J+m-1)]\alpha^\dagger_1\alpha^\dagger_2\left(a^\dagger\right)^{J-(m-1)-1}|0>. \end{aligned} \tag{4.40}$$

This becomes a finite-dimensional representation of the even subalgebra if one requires that the r.h.s equal to $(J+m)|J,m-1;q>$. This requirement is satisfied if $c_{J,q}=2J$ and $\bar{c}_{J,q}=q$. So representations of $osp(2|2)$ are spanned by the $su(2)\oplus u(1)$ multiplets:

$$\begin{aligned} &|J,m;q>=\left[2J\left(a^\dagger\right)^{J-m}+q(J-m)\alpha^\dagger_1\alpha^\dagger_2\left(a^\dagger\right)^{J-m-1}\right]|0>,\quad m=J,J-1,\cdots,-J,\\ &|J-\frac{1}{2},m;q-\frac{1}{2}>=\alpha^\dagger_1\left(a^\dagger\right)^{J-1/2-m}|0>,\quad m=J-\frac{1}{2},J-\frac{3}{2},\cdots,-(J-\frac{1}{2}),\\ &|J-\frac{1}{2},m;q+\frac{1}{2}>=\alpha^\dagger_2\left(a^\dagger\right)^{J-1/2-m}|0>,\quad m=J-\frac{1}{2},J-\frac{3}{2},\cdots,-(J-\frac{1}{2}),\\ &|J-1,m;q>=\alpha^\dagger_1\alpha^\dagger_2\left(a^\dagger\right)^{J-1-m}|0>,\quad m=J-1,J-2,\cdots,-(J-1). \end{aligned} \tag{4.41}$$

Here $J\geq 1/2$ in the first three expressions and $J\geq 1$ in the last one. Note that the trivial 1-dimensional representation of $osp(2|2)$ (corresponding to $J=0=q$) is given by the vacuum $|0>$. It follows that in the non-standard basis, $osp(2|2)\downarrow su(2)\oplus u(1)$ branching rule is

$$\pi_{(J,q)}=\sigma_{(J,q)}\oplus\sigma_{(J-1/2,q-1/2)}\oplus\sigma_{(J-1/2,q+1/2)}\oplus\sigma_{(J-1,q)} \tag{4.42}$$

for $q\neq\mp J$. Here $\pi_{(J,q)}$ stands for a representation of $osp(1|2)$ (in the non-standard basis) labelled by (J,q), and $\sigma_{(J,q)}$ etc stand for the four $su(2)\oplus u(1)$ multiplets in (4.41).

By means of (3.29) and (4.41) it can be shown that the actions of the odd generators on these $su(2)\oplus u(1)$ multiplets are given by

$$
\begin{aligned}
\Gamma(e)|J,m;q> &= (J-m)(q+J)|J-\frac{1}{2},m+\frac{1}{2};q+\frac{1}{2}>,\\
\Gamma(f)|J,m;q> &= (J+m)(q-J)|J-\frac{1}{2},m-\frac{1}{2};q-\frac{1}{2}>,\\
\Gamma(\bar{e})|J,m;q> &= -(J-m)(q-J)|J-\frac{1}{2},m+\frac{1}{2};q-\frac{1}{2}>,\\
\Gamma(\bar{f})|J,m;q> &= -(J+m)(q+J)|J-\frac{1}{2},m-\frac{1}{2};q+\frac{1}{2}>,
\end{aligned}
\tag{4.43}
$$

$$
\begin{aligned}
\Gamma(\bar{e})|J-\frac{1}{2},m;q-\frac{1}{2}> &= 0,\\
\Gamma(\bar{f})|J-\frac{1}{2},m;q-\frac{1}{2}> &= \frac{1}{2J}|J,m-\frac{1}{2};q> -\frac{J-1/2+m}{2J}(q+J)|J-1,m-\frac{1}{2};q>,\\
\Gamma(e)|J-\frac{1}{2},m;q-\frac{1}{2}> &= \frac{1}{2J}|J,m+\frac{1}{2};q> -\frac{J-1/2-m}{2J}(q+J)|J-1,m+\frac{1}{2};q>,\\
\Gamma(f)|J-\frac{1}{2},m;q-\frac{1}{2}> &= 0,
\end{aligned}
\tag{4.44}
$$

$$
\begin{aligned}
\Gamma(e)|J-\frac{1}{2},m;q+\frac{1}{2}> &= 0,\\
\Gamma(f)|J-\frac{1}{2},m;q+\frac{1}{2}> &= \frac{1}{2J}|J,m-\frac{1}{2};q> +\frac{J-1/2+m}{2J}(q-J)|J-1,m-\frac{1}{2};q>,\\
\Gamma(\bar{e})|J-\frac{1}{2},m;q+\frac{1}{2}> &= \frac{1}{2J}|J,m+\frac{1}{2};q> -\frac{J-1/2-m}{2J}(q-J)|J-1,m+\frac{1}{2};q>,\\
\Gamma(\bar{f})|J-\frac{1}{2},m;q+\frac{1}{2}> &= 0,
\end{aligned}
\tag{4.45}
$$

and

$$
\begin{aligned}
\Gamma(e)|J-1,m;q> &= |J-\frac{1}{2},m+\frac{1}{2};q+\frac{1}{2}>,\\
\Gamma(f)|J-1,m;q> &= -|J-\frac{1}{2},m-\frac{1}{2};q-\frac{1}{2}>,\\
\Gamma(\bar{e})|J-1,m;q> &= -|J-\frac{1}{2},m+\frac{1}{2};q-\frac{1}{2}>,\\
\Gamma(\bar{f})|J-1,m;q> &= |J-\frac{1}{2},m-\frac{1}{2};q+\frac{1}{2}>,
\end{aligned}
\tag{4.46}
$$

Note that the multiplet $|J,m;q>$ has dimension $2J+1$, both $|J-\frac{1}{2},m;q\mp\frac{1}{2}>$ have dimension $2J$ and the dimension of $|J-1,m;q>$ is $2J-1$. So for $q\neq\mp J$, they constitute irreducible typical representation of dimension $8J$ of $osp(2|2)$.

When $q = \mp J$, the representations become atypical. We have different types of atypical representations. The Casimirs for all such representations vanish, and yet they are not the trivial one-dimensional representation. One type is obtained by applying the odd generators to the $su(2) \oplus u(1)$ representation $|J, m; q>$. For $q = J$, the multiplets $|J - \frac{1}{2}, m; q - \frac{1}{2}>$ and $|J-1, m; q>$ do not appear, and only $|J, m; q>$ and $|J - \frac{1}{2}, m; q + \frac{1}{2}>$ survive. They form irreducible atypical representation of dimension $4J+1$. Similarly, for $q = -J$, the multiplets $|J - \frac{1}{2}, m; q + \frac{1}{2}>$ and $|J-1, m; q>$ do not appear, and only $|J, m; q>$ and $|J - \frac{1}{2}, m; q - \frac{1}{2}>$ remain. They also form irreducible atypical representation of dimension $4J+1$. Other types of atypical representations are not irreducible and are obatined by applying odd generators to the $su(2) \oplus u(1)$ representations $|J - \frac{1}{2}, m; q \mp \frac{1}{2}>$. In both cases, the representation contains all multiplets and thus has dimension $8J$. These representations are not fully reducible.

4.2 Representations in the Standard Basis

Representations of $osp(2|2)$ in the standard basis are labelled by (p, q) with p being a non-negative integer or half-integer and q any complex number.

In the standard basis, the four independent combinations of creation operators acting on $|0>$ are

$$\begin{aligned}
&\left(a^\dagger\right)^{p-m}|0>, \quad p-m \in \mathbf{Z}_+,\\
&\alpha_1^\dagger \left(a^\dagger\right)^{p-m-1/2}|0>, \quad p-\frac{1}{2}-m \in \mathbf{Z}_+,\\
&\alpha_2^\dagger \left(a^\dagger\right)^{p-m-3/2}|0>, \quad p-\frac{3}{2}-m \in \mathbf{Z}_+,\\
&\alpha_1^\dagger\alpha_2^\dagger \left(a^\dagger\right)^{p-m-2}|0>, \quad p-2-m \in \mathbf{Z}_+.
\end{aligned} \tag{4.47}$$

So again each $osp(2|2)$ representation decomposes into at most four representations of the even subalgebra $su(2) \oplus u(1)$. To construct the four multiplets, we note that the first and the last states are already representations of $su(2) \oplus u(1)$ with highest weight weights (p, q) and $(p, q+1)$, respectively. We denote these two multiplets by $|p, m; q>$ and $|p, m; q+1>$, respectively. We can combine the second and the third states into two independent multiplets of $su(2) \oplus u(1)$ with highest weights $(p - \frac{1}{2}, q + \frac{1}{2})$ and $(p + \frac{1}{2}, q + \frac{1}{2})$, respectively. This is seen as follows. Let

$$\chi^m_{p,q} = \frac{1}{2} c^m_{p,q} \alpha_1^\dagger \left(a^\dagger\right)^{p-m-1/2}|0> + \bar{c}^m_{p,q} \alpha_2^\dagger \left(a^\dagger\right)^{p-m-3/2}|0>, \tag{4.48}$$

where $c^m_{p,q}$ and $\bar{c}^m_{p,q}$ are functions of p, q, m to be determined. Then,

$$\begin{aligned}
\Gamma(E)\chi^m_{p,q} \;=\;& \frac{1}{2}\left((p-m-\frac{1}{2})c^m_{p,q} - \bar{c}^m_{p,q}\right) \alpha_1^\dagger \left(a^\dagger\right)^{p-m-3/2}|0>\\
&+(p-m-\frac{3}{2})\bar{c}^m_{p,q}\alpha_2^\dagger \left(a^\dagger\right)^{p-m-5/2}|0> .
\end{aligned} \tag{4.49}$$

To make the representation finite-dimensional, the r.h.s. of this equation must equal to $(p-m-\frac{x}{2})\chi^{m+1}_{p,q}$ for some integer x. On the other hand,

$$\begin{aligned}\Gamma(F)\chi^m_{p,q} &= \frac{1}{2}\left((p+m+1)c^m_{p,q}-\frac{1}{2}\bar{c}^m_{p,q}\right)\alpha^\dagger_1\left(a^\dagger\right)^{p-m+1/2}|0> \\ &\quad \left((p+m+1)\bar{c}^m_{p,q}-\frac{1}{2}c^m_{p,q}\right)\alpha^\dagger_2\left(a^\dagger\right)^{p-m-1/2}|0> . \end{aligned} \tag{4.50}$$

This must equal to $(p+m+\frac{y}{2})\chi^{m-1}_{p,q}$ for some integer y in order for the representation to be finite-dimensional. These two requirements are satisfied if [21] either

$$x=1,\quad y=3,\quad c^m_{p,q}=3p+m+\frac{5}{2},\quad \bar{c}^m_{p,q}=-(p-m-\frac{1}{2}) \tag{4.51}$$

or

$$x=3,\quad y=1,\quad c^m_{p,q}=\bar{c}^m_{p,q}=1. \tag{4.52}$$

Also it is easily seen that

$$\Gamma(H)\chi^m_{p,q}=2(m+1)\chi^m_{p,q},\quad \Gamma(H')\chi^m_{p,q}=2(q+\frac{1}{2})\chi^m_{p,q}. \tag{4.53}$$

It follows that $\chi^m_{p,q}$ has highest weight $(p+1/2,q+1/2)$ for the first case (4.51) (where $m_{\max}=p-1/2$) and highest weight $(p-1/2,q+1/2)$ for the second case (4.52) (where $m_{\max}=p-3/2$). This justifies the use of notation, $|p+\frac{1}{2},m;q+\frac{1}{2}>$ and $|p-\frac{1}{2},m;q+\frac{1}{2}>$, for these two multiplets, respectively.

Therefore, we have the following four $su(2)\oplus u(1)$ multiplets which span finite-dimensional representations of $osp(2|2)$:

$$\begin{aligned}|p,m;q> &= \left(a^\dagger\right)^{p-m}|0>,\quad m=p,p-1,\cdots,-p,\quad p\geq 0,\\ |p-\frac{1}{2},m;q+\frac{1}{2}> &= \left(\alpha^\dagger_2+\frac{1}{2}\alpha^\dagger_1 a^\dagger\right)\left(a^\dagger\right)^{p-3/2-m}|0>,\\ &\quad m=p-\frac{3}{2},p-\frac{5}{2},\cdots,-(p+\frac{1}{2}),\quad p\geq\frac{1}{2},\\ |p+\frac{1}{2},m;q+\frac{1}{2}> &= \left(p+m+\frac{3}{2}\right)\alpha^\dagger_1\left(a^\dagger\right)^{p-1/2-m}|0>\\ &\quad -\left(p-m-\frac{1}{2}\right)\left(\alpha^\dagger_2-\frac{1}{2}\alpha^\dagger_1 a^\dagger\right)\left(a^\dagger\right)^{p-m-3/2}|0>,\\ &\quad m=p-\frac{1}{2},p-\frac{3}{2},\cdots,-(p+\frac{3}{2}),\quad p\geq 0,\\ |p,m;q+1> &= \alpha^\dagger_1\alpha^\dagger_2\left(a^\dagger\right)^{p-2-m}|0>,\\ &\quad m=p-2,p-3,\cdots,-(p+2),\quad p\geq 0.\end{aligned} \tag{4.54}$$

We remark that the trivial 1-dimensional representation (for which $p = 0 = q$) is provided by $|0>$. The $osp(2|2) \downarrow su(2) \oplus u(1)$ branching rule in the standard basis is given by

$$\pi_{(p,q)} = \sigma_{(p,q)} \oplus \sigma_{(p-1/2,q+1/2)} \oplus \sigma_{(p+1/2,q+1/2)} \oplus \sigma_{(p,q+1)} \tag{4.55}$$

for $q \neq p, -p-1$. Here $\pi_{(p,q)}$ is a representation of $osp(2|2)$ (in the standard basis) labelled by (p, q), and $\sigma_{(p,q)}$ etc are the four $su(2) \oplus u(1)$ multiplets in (4.54).

Using (3.35) and (4.54), one can compute the actions of the odd generators on these $su(2) \oplus u(1)$ multiplets. The result is given by [21]

$$\begin{aligned}
\Gamma(e)|p,m;q> &= 0,\\
\Gamma(f)|p,m;q> &= \frac{p-m}{2p+1}(q+p+1)|p-\frac{1}{2},m-\frac{1}{2};q+\frac{1}{2}>\\
&\quad +\frac{1}{2p+1}(q-p)|p+\frac{1}{2},m-\frac{1}{2};q+\frac{1}{2}>,\\
\Gamma(\bar{e})|p,m;q> &= 0,\\
\Gamma(\bar{f})|p,m;q> &= -\frac{p+m}{2p+1}(q+p+1)|p-\frac{1}{2},m-\frac{3}{2};q+\frac{1}{2}>,\\
&\quad +\frac{1}{2p+1}(q-p)|p+\frac{1}{2},m-\frac{3}{2};q+\frac{1}{2}>,
\end{aligned} \tag{4.56}$$

$$\begin{aligned}
\Gamma(e)|p-\frac{1}{2},m;q+\frac{1}{2}> &= |p,m+\frac{1}{2};q>,\\
\Gamma(f)|p-\frac{1}{2},m;q+\frac{1}{2}> &= (q-p)|p,m-\frac{1}{2};q+1>,\\
\Gamma(\bar{e})|p-\frac{1}{2},m;q+\frac{1}{2}> &= |p,m+\frac{3}{2};q>,\\
\Gamma(\bar{f})|p-\frac{1}{2},m;q+\frac{1}{2}> &= (q-p)|p,m-\frac{3}{2};q+1>,
\end{aligned} \tag{4.57}$$

$$\begin{aligned}
\Gamma(e)|p+\frac{1}{2},m;q+\frac{1}{2}> &= (p+m+\frac{3}{2})|p,m+\frac{1}{2};q>,\\
\Gamma(f)|p+\frac{1}{2},m;q+\frac{1}{2}> &= -(p-m-\frac{1}{2})(q+p+1)|p,m-\frac{1}{2};q+1>,\\
\Gamma(\bar{e})|p+\frac{1}{2},m;q+\frac{1}{2}> &= -(p-m-\frac{1}{2})|p,m+\frac{3}{2};q>,\\
\Gamma(\bar{f})|p+\frac{1}{2},m;q+\frac{1}{2}> &= (p+m+\frac{3}{2})(q+p+1)|p,m-\frac{3}{2};q+1>,
\end{aligned} \tag{4.58}$$

and

$$\Gamma(e)|p,m;q+1> = \frac{p+m+2}{2p+1}|p-\frac{1}{2},m+\frac{1}{2};q+\frac{1}{2}>$$

$$
\begin{aligned}
& & & -\frac{1}{2p+1}|p+\frac{1}{2},m+\frac{1}{2};q+\frac{1}{2}>, \\
\Gamma(f)|p,m;q+1> & = & & 0, \\
\Gamma(\bar{e})|p,m;q+1> & = & & -\frac{p-m-2}{2p+1}|p-\frac{1}{2},m+\frac{3}{2};q+\frac{1}{2}> \\
& & & -\frac{1}{2p+1}|p+\frac{1}{2},m+\frac{3}{2};q+\frac{1}{2}>, \\
\Gamma(\bar{f})|p,m;q+1> & = & & 0, \qquad (4.59)
\end{aligned}
$$

Note that both $|p,m;q>$ and $|p,m;q+1>$ have dimension $2p+1$, $|p-\frac{1}{2},m;q+\frac{1}{2}>$ has dimension $2p$ and the dimension of $|p+\frac{1}{2},m;q+\frac{1}{2}>$ is $2p+2$. So for $q \neq p, -p-1$, they constitute irreducible typical representation of dimension $8p+4$ of $osp(2|2)$.

When $q=p, \ -p-1$, the representations become atypical. The Casimirs for such representations vanish. For $q=p$, one drops $|p+\frac{1}{2},m;q+\frac{1}{2}>$ and $|p,m;q+1>$ and only keeps $|p,m;q>$ and $|p-\frac{1}{2},m;q+\frac{1}{2}>$ survive. They form irreducible atypical representation of $osp(2|2)$ of dimension $4p+1$ ($p \geq 1/2$). For $q=-p-1$, $|p-\frac{1}{2},m;q+\frac{1}{2}>$ and $|p,m;q+1>$ disappear and only $|p,m;q>$ and $|p+\frac{1}{2},m;q+\frac{1}{2}>$ remain. They constitute irreducible atypical representation of dimension $4p+3$. Other atypical representations are not irreducible and obtained by retaining all the $su(2)\oplus u(1)$ multiplets. Such representation has dimension $8p+4$ and is not fully reducible.

5 Boson-Fermion Realizations of $gl(2|2)$ and Its Subalgebras

In this section, we obtain boson-fermion realizations of the superalgebra $gl(2|2)$ and its subalgebras in the non-standard basis.

This superalgebra is non-semisimple and can be written as $gl(2|2)^{\rm even} \oplus gl(2|2)^{\rm odd}$, where

$$
\begin{aligned}
gl(2|2)^{\rm even} & = & & u(1)\oplus gl(2)\oplus gl(2) \\
& = & & \{I\}\oplus\{\{E_{12},E_{21},H_1\},\{E_{34},E_{43},H_2\},N\}, \\
gl(2|2)^{\rm odd} & = & & \{E_{13},E_{31},E_{32},E_{23},E_{24},E_{42},E_{14},E_{41}\}. \qquad (5.60)
\end{aligned}
$$

In the non-standard basis, E_{13},E_{32},E_{24} (E_{31},E_{23},E_{42}) are simple raising (lowering) generators, E_{12},E_{34},E_{14} (E_{21},E_{43},E_{41}) are non-simple raising (lowering) generators and I,H_1,H_2,N are elements of the Cartan subalgebra. We have

$$
\begin{aligned}
H_1 & = & & E_{11}-E_{22}, \quad H_2=E_{33}-E_{44}, \\
I & = & & E_{11}+E_{22}+E_{33}+E_{44}, \\
N & = & & E_{11}+E_{22}-E_{33}-E_{44}+\beta I \qquad (5.61)
\end{aligned}
$$

with β being an arbitrary parameter. That N is not uniquely determined is a consequence of the fact that $gl(2|2)$ is non-semisimple. The generators obey the following (anti-)commutation relations:

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - (-1)^{([i]+[j])([k]+[l])} \delta_{il} E_{kj}, \tag{5.62}$$

where $[E_{ij}, E_{kl}] \equiv E_{ij}E_{kl} - (-1)^{([i]+[j])([k]+[l])} E_{kl}E_{ij}$ is a commutator or an anticommutator, $[1] = [2] = 0,\ [3] = [4] = 1$, and $E_{ii},\ i = 1, 2, 3, 4$ are related to I, H_1, H_2, N via (5.61).

Let $|hw>$ be the highest weight state of $gl(2|2)$ defined by

$$\begin{aligned}
&H_1|hw> = 2J_1|hw>, \quad H_2|hw> = 2J_2|hw>, \\
&I|hw> = 2\omega|hw>, \quad N|hw> = 2\lambda|hw>, \\
&E_{13}|hw> = E_{32}|hw> = E_{24}|hw> = E_{14}|hw> = E_{12}|hw> = E_{34}|hw> = 0.
\end{aligned} \tag{5.63}$$

Then state vectors are mapped into functions

$$\begin{aligned}
\psi_{J_1,J_2,\omega,\lambda} &= <hw|e^{\alpha^\dagger_{13}E_{13}+\alpha^\dagger_{32}E_{32}+\alpha^\dagger_{24}E_{24}+\alpha^\dagger_{14}E_{14}+a^\dagger_{12}E_{12}+a^\dagger_{34}E_{34}}|\psi>|0> \\
&= <hw|e^{\alpha^\dagger_{13}E_{13}}e^{\alpha^\dagger_{32}E_{32}}e^{\alpha^\dagger_{24}E_{24}}e^{(a^\dagger_{12}+\frac{1}{2}\alpha^\dagger_{13}\alpha^\dagger_{32})E_{12}} \\
&\quad \times e^{(a^\dagger_{34}+\frac{1}{2}\alpha^\dagger_{32}\alpha^\dagger_{24})E_{34}}e^{(\alpha^\dagger_{14}+\frac{1}{2}a^\dagger_{12}\alpha^\dagger_{24}-\frac{1}{2}\alpha^\dagger_{13}a^\dagger_{34}+\frac{1}{3}\alpha^\dagger_{13}\alpha^\dagger_{32}\alpha^\dagger_{24})E_{14}}|\psi>|0>,
\end{aligned} \tag{5.64}$$

where the Zassenhaus formula (2.4) has been used repeatedly, and operators A are mapped as before:

$$\begin{aligned}
A|\psi> \longrightarrow \Gamma(A)\psi_{J_1,J_2,\omega,\lambda} &= <hw|e^{\alpha^\dagger_{13}E_{13}}e^{\alpha^\dagger_{32}E_{32}}e^{\alpha^\dagger_{24}E_{24}} \\
&\quad \times e^{(a^\dagger_{12}+\frac{1}{2}\alpha^\dagger_{13}\alpha^\dagger_{32})E_{12}}e^{(a^\dagger_{34}+\frac{1}{2}\alpha^\dagger_{32}\alpha^\dagger_{24})E_{34}} \\
&\quad \times e^{(\alpha^\dagger_{14}+\frac{1}{2}a^\dagger_{12}\alpha^\dagger_{24}-\frac{1}{2}\alpha^\dagger_{13}a^\dagger_{34}+\frac{1}{3}\alpha^\dagger_{13}\alpha^\dagger_{32}\alpha^\dagger_{24})E_{14}}A|\psi>|0>.
\end{aligned} \tag{5.65}$$

Here $\alpha^\dagger_{ij}$ (α_{ij}) are fermion operators with number operators $N_{\alpha_{ij}}$ and $a^\dagger_{ij}$ (a_{ij}) are boson operators with number operators $N_{a_{ij}}$. They obey relations

$$\begin{aligned}
&\{\alpha_{ij}, \alpha^\dagger_{kl}\} = \delta_{ik}\delta_{jl}, \quad (\alpha_{ij})^2 = \left(\alpha^\dagger_{ij}\right)^2 = 0, \\
&[N_{\alpha_{ij}}, \alpha_{kl}] = -\delta_{ik}\delta_{jl}\alpha_{kl}, \quad [N_{\alpha_{ij}}, \alpha^\dagger_{kl}] = \delta_{ik}\delta_{jl}\alpha^\dagger_{kl}, \\
&[a_{ij}, a^\dagger_{kl}] = \delta_{ik}\delta_{jl}, \\
&[N_{a_{ij}}, a_{kl}] = -\delta_{ik}\delta_{jl}a_{kl}, \quad [N_{a_{ij}}, a^\dagger_{kl}] = \delta_{ik}\delta_{jl}a^\dagger_{kl},
\end{aligned} \tag{5.66}$$

and all other (anti-)commutators vanish. Moreover, $\alpha_{13}|0> = \alpha_{32}|0> = \alpha_{24}|0> = \alpha_{14}|0> = a_{12}|0> = a_{34}|0> = 0$.

Taking E_{13}, E_{31} etc in turn and after long but straightforward computations, we find

$$
\begin{aligned}
E_{13} &= \alpha_{13} + \frac{1}{2}\alpha_{32}^{\dagger}a_{12} - \frac{1}{2}\left(a_{34}^{\dagger} + \frac{1}{6}\alpha_{32}^{\dagger}\alpha_{24}^{\dagger}\right)\alpha_{14},\\
E_{31} &= (\omega + J_1 + J_2)\alpha_{13}^{\dagger} - a_{12}^{\dagger}\alpha_{32} + \alpha_{14}^{\dagger}a_{34} - \frac{1}{2}\alpha_{13}^{\dagger}\left(N_{\alpha_{32}} + N_{a_{12}} + N_{a_{34}} + N_{\alpha_{14}}\right)\\
&\quad -\frac{1}{12}\alpha_{13}^{\dagger}\alpha_{24}^{\dagger}\left(\alpha_{32}^{\dagger}a_{34} + a_{12}^{\dagger}\alpha_{14}\right),\\
E_{32} &= \alpha_{32} + \frac{1}{2}\alpha_{13}^{\dagger}a_{12} + \frac{1}{2}\alpha_{24}^{\dagger}\left(a_{34} + \frac{1}{3}\alpha_{13}^{\dagger}\alpha_{14}\right),\\
E_{23} &= (\omega - J_1 + J_2)\alpha_{32}^{\dagger} + a_{12}^{\dagger}\alpha_{13} - a_{34}^{\dagger}\alpha_{24} + \frac{1}{2}\alpha_{32}^{\dagger}\left(N_{\alpha_{13}} - N_{\alpha_{24}} + N_{a_{12}} - N_{a_{34}}\right)\\
&\quad +\frac{1}{6}\alpha_{32}^{\dagger}\left(\alpha_{13}^{\dagger}a_{34}^{\dagger} + a_{12}^{\dagger}\alpha_{24}^{\dagger}\right)\alpha_{14},\\
E_{24} &= \alpha_{24} + \frac{1}{2}\alpha_{32}^{\dagger}a_{34} + \frac{1}{2}\left(a_{12}^{\dagger} - \frac{1}{6}\alpha_{13}^{\dagger}\alpha_{32}^{\dagger}\right)\alpha_{14},\\
E_{42} &= (\omega - J_1 - J_2)\alpha_{24}^{\dagger} + a_{34}^{\dagger}\alpha_{32} + \alpha_{14}^{\dagger}a_{12} + \frac{1}{2}\alpha_{24}^{\dagger}\left(N_{\alpha_{32}} + N_{a_{12}} + N_{a_{34}} + N_{\alpha_{14}}\right)\\
&\quad +\frac{1}{12}\alpha_{13}^{\dagger}\alpha_{24}^{\dagger}\left(a_{34}^{\dagger}\alpha_{14} - \alpha_{32}^{\dagger}a_{12}\right),\\
H_1 &= 2J_1 - 2N_{a_{12}} - N_{\alpha_{13}} - N_{\alpha_{32}} + N_{\alpha_{24}} - N_{\alpha_{14}},\\
H_2 &= 2J_2 - 2N_{a_{34}} + N_{\alpha_{13}} - N_{\alpha_{32}} - N_{\alpha_{24}} - N_{\alpha_{14}},\\
I &= 2\omega,\\
N &= 2\lambda - 2\left(N_{\alpha_{13}} - N_{\alpha_{32}} + N_{\alpha_{24}} + N_{\alpha_{14}}\right),\\
E_{12} &= a_{12} - \frac{1}{2}\alpha_{24}^{\dagger}\alpha_{14},\\
E_{21} &= 2J_1a_{12}^{\dagger} - (\omega + J_2)\alpha_{13}^{\dagger}\alpha_{32}^{\dagger} - a_{12}^{\dagger}\left(N_{\alpha_{13}} + N_{\alpha_{32}} + N_{a_{12}}\right)\\
&\quad -\left(\alpha_{14}^{\dagger} - \frac{1}{2}\alpha_{13}^{\dagger}a_{34}^{\dagger}\right)\alpha_{24} + \frac{1}{2}\left(a_{12}^{\dagger} + \frac{1}{3}\alpha_{13}^{\dagger}\alpha_{32}^{\dagger}\right)N_{\alpha_{24}}\\
&\quad +\frac{1}{2}\alpha_{32}^{\dagger}\left(\alpha_{14}^{\dagger} + \frac{1}{2}a_{12}^{\dagger}\alpha_{24}^{\dagger}\right)a_{34} + \frac{1}{4}\alpha_{13}^{\dagger}\alpha_{32}^{\dagger}N_{a_{34}}\\
&\quad -\frac{1}{2}\left(a_{12}^{\dagger} - \frac{1}{6}\alpha_{13}^{\dagger}\alpha_{32}^{\dagger}\right)N_{\alpha_{14}} - \frac{1}{4}\left(a_{12}^{\dagger}\alpha_{24}^{\dagger} + \alpha_{13}^{\dagger}a_{34}^{\dagger} + \frac{1}{6}\alpha_{13}^{\dagger}\alpha_{32}^{\dagger}\alpha_{24}^{\dagger}\right)a_{12}^{\dagger}\alpha_{14},\\
E_{34} &= a_{34} + \frac{1}{2}\alpha_{13}^{\dagger}\alpha_{14},\\
E_{43} &= 2J_2a_{34}^{\dagger} - (\omega - J_1)\alpha_{32}^{\dagger}\alpha_{24}^{\dagger} - a_{34}^{\dagger}\left(N_{\alpha_{32}} + N_{\alpha_{24}} + N_{a_{34}}\right)\\
&\quad +\left(\alpha_{14}^{\dagger} + \frac{1}{2}a_{12}^{\dagger}\alpha_{24}^{\dagger}\right)\alpha_{13} + \frac{1}{2}\left(a_{34}^{\dagger} - \frac{1}{3}\alpha_{32}^{\dagger}\alpha_{24}^{\dagger}\right)N_{\alpha_{13}}\\
&\quad -\frac{1}{2}\alpha_{32}^{\dagger}\left(\alpha_{14}^{\dagger} - \frac{1}{2}\alpha_{13}^{\dagger}a_{34}^{\dagger}\right)a_{12} - \frac{1}{4}\alpha_{32}^{\dagger}\alpha_{24}^{\dagger}N_{a_{12}} - \frac{1}{2}\left(a_{34}^{\dagger} + \frac{1}{6}\alpha_{32}^{\dagger}\alpha_{24}^{\dagger}\right)N_{\alpha_{14}}\\
&\quad +\frac{1}{4}\left(a_{12}^{\dagger}\alpha_{24}^{\dagger} + \alpha_{13}^{\dagger}a_{34}^{\dagger} - \frac{1}{6}\alpha_{13}^{\dagger}\alpha_{32}^{\dagger}\alpha_{24}^{\dagger}\right)a_{34}^{\dagger}\alpha_{14},
\end{aligned}
$$

$$
\begin{aligned}
E_{14} &= \alpha_{14}, \\
E_{41} &= (\omega+J_1-J_2)\alpha_{14}^\dagger+\frac{1}{2}(\omega+J_1+3J_2)a_{12}^\dagger\alpha_{24}^\dagger \\
&\quad +\frac{1}{2}(-\omega+3J_1+J_2)\alpha_{13}^\dagger a_{34}^\dagger-\frac{1}{3}(J_1-J_2)\alpha_{13}^\dagger\alpha_{32}^\dagger\alpha_{24}^\dagger \\
&\quad -a_{12}^\dagger a_{34}^\dagger\left(\alpha_{32}-\frac{1}{6}\alpha_{13}^\dagger\alpha_{24}^\dagger\alpha_{14}\right)-\left(\alpha_{14}^\dagger+\frac{1}{2}a_{12}^\dagger\alpha_{24}^\dagger\right)N_{\alpha_{13}} \\
&\quad -\frac{1}{2}\left(\alpha_{13}^\dagger a_{34}^\dagger+a_{12}^\dagger\alpha_{24}^\dagger\right)N_{\alpha_{32}}+\left(\alpha_{14}^\dagger-\frac{1}{2}\alpha_{13}^\dagger a_{34}^\dagger\right)N_{\alpha_{24}} \\
&\quad -\left(\alpha_{14}^\dagger+\frac{1}{2}a_{12}^\dagger\alpha_{24}^\dagger+\frac{1}{12}\alpha_{13}^\dagger\alpha_{32}^\dagger\alpha_{24}^\dagger\right)N_{a_{12}} \\
&\quad +\left(\alpha_{14}^\dagger-\frac{1}{2}\alpha_{13}^\dagger a_{34}^\dagger+\frac{1}{12}\alpha_{13}^\dagger\alpha_{32}^\dagger\alpha_{24}^\dagger\right)N_{a_{34}}. \qquad (5.67)
\end{aligned}
$$

This gives a boson-fermion realization of $gl(2|2)$ in the non-standard basis. This result reconfirms the differential operator realization found in [22]. Here we have corrected a few misprints appeared in the expressions for $d_{E_{31}}, d_{E_{21}}, d_{E_{43}}$ and $d_{E_{41}}$ in the eq.(3.3) of [22].

From the above realization, we may deduce realizations of subalgebras $gl(2|1)$, $gl(1|2)$ and $gl(1|1)$ with less numbers of bosons and fermions. For the subalgebra $gl(2|1)$, which is generated by $\{I, E_{13}, E_{31}, E_{32}, E_{23}, E_{12}, E_{21}, H_1, N\}$, we choose $a_{34}=a_{34}^\dagger=\alpha_{24}=\alpha_{24}^\dagger=\alpha_{14}=\alpha_{14}^\dagger=0$ to obtain

$$
\begin{aligned}
E_{13} &= \alpha_{13}+\frac{1}{2}\alpha_{32}^\dagger a_{12}, \\
E_{31} &= (\omega+J_1)\alpha_{13}^\dagger-a_{12}^\dagger\alpha_{32}-\frac{1}{2}\alpha_{13}^\dagger\left(N_{\alpha_{32}}+N_{a_{12}}\right), \\
E_{32} &= \alpha_{32}+\frac{1}{2}\alpha_{13}^\dagger a_{12}, \\
E_{23} &= (\omega-J_1)\alpha_{32}^\dagger+a_{12}^\dagger\alpha_{13}+\frac{1}{2}\alpha_{32}^\dagger\left(N_{\alpha_{13}}+N_{a_{12}}\right), \\
H_1 &= 2J_1-2N_{a_{12}}-N_{\alpha_{13}}-N_{\alpha_{32}}, \\
I &= 2\omega, \\
N &= 2\lambda-2\left(N_{\alpha_{13}}-N_{\alpha_{32}}\right), \\
E_{12} &= a_{12}, \\
E_{21} &= 2J_1a_{12}^\dagger-\omega\alpha_{13}^\dagger\alpha_{32}^\dagger-a_{12}^\dagger\left(N_{\alpha_{13}}+N_{\alpha_{32}}+N_{a_{12}}\right). \qquad (5.68)
\end{aligned}
$$

Similarly we may deduce a realization of the subalgebra $gl(1|2)$, which is generated by $\{I, E_{32}, E_{23}, E_{24}, E_{42}, E_{34}, E_{43}, H_2, N\}$, by choosing $a_{12}=a_{12}^\dagger=\alpha_{13}=\alpha_{13}^\dagger=\alpha_{14}=\alpha_{14}^\dagger=0$. Finally, the subalgebra $gl(1|1)$ is generated by $\{e\equiv E_{13}+E_{24}, f\equiv E_{31}+E_{42}, I, N\}$. Its realization is deduced by choosing $a_{12}=a_{12}^\dagger=a_{34}=a_{34}^\dagger=\alpha_{32}=\alpha_{32}^\dagger=\alpha_{14}=\alpha_{14}^\dagger=0$. We have

$$
e=\alpha_{13}+\alpha_{24}, \quad f=2\omega(\alpha_{13}^\dagger+\alpha_{24}^\dagger), \quad I=2\omega, \quad N=2\lambda-2(N_{\alpha_{13}}+N_{\alpha_{24}}). \qquad (5.69)
$$

Writing $\alpha \equiv \alpha_{13} + \alpha_{24}$ (and thus $\alpha^\dagger \equiv \alpha^\dagger_{13} + \alpha^\dagger_{24}$, $N_\alpha \equiv N_{\alpha_{13}} + N_{\alpha_{24}}$), we see that $gl(1|1)$ is realized in terms of one fermion $\alpha, \alpha^\dagger$.

6 Conclusions and Discussions

In this article we have discussed the application of the super coherent state method to the construction of free boson-fermion realizations and representations of superalgebras. We have worked through a few examples corresponding to lower rank superalgebras. The representations are constructed out of particle states in the super-Fock spaces.

As the results show, the computations become very involved with the increase of the rank of superalgebras concerned. Nevertheless, it is expected that the procedure can be extended to higher rank superalgebras so as to obtain free boson-fermion realizations and explicitly construct representations of the superalgebras with the help of the realizations. In the previous section we have obtained the free boson-fermion realization of the rank 3 non-semisimple superalgebra $gl(2|2)$ in the non-standard basis. This superalgebra has applications in disordered systems and integer quantum Hall plateau transition [23]. It is interesting to explicitly construct its finite-dimensional representations by the method described in this paper. This then will enable the construction of primary fields of the $gl(2|2)$ non-unitary CFT. This is a problem under investigation.

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Note added in proof:

All finite-dimensional typical and atypical representations of $gl(2|2)$ and corresponding primary fields of the $gl(2|2)$ non-unitary CFT in the standard basis have recently been constructed by us in [24, 25].

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